Name: Enrolment No:	

UPES

End Semester Examination, Dec 2024

Programme Name : Integrated B.Sc.-M.Sc. Mathematics **Semester : VII**
Course Name : Approximation Theory **Time : 03 hrs**
Course Code : MATH 4005P **Max. Marks: 100**
Nos. of page(s) : 02

Instructions: All questions are compulsory. There is an internal choice in Q9 and Q11.

SECTION A

Answer all questions

S. No.		Marks	CO
Q 1	State the theorem on existence of best approximation in a metric space.	**4**	**CO1**
Q 2	Does there exist a best approximation for $\alpha = \sqrt{1.4}$ in some subset S of real line described as $S = \{x \in \mathbb{Q}: 2 \leq x^2 \leq 3\}$? Justify your answer.	**4**	**CO1**
Q 3	Prove that the norm defined as $\|(x_1, x_2)\|_\infty = \max\{\|x_1\|, \|x_2\|\}$ is not strictly convex on the normed linear space $\mathbb{R}^2$.	**4**	**CO2**
Q 4	Approximate $f(x) = \sin x$ in the interval $[0, \pi]$ by some polygonal function $p(x)$ such that $p(x_i) = f(x_i)$ for $x_i = 0, \frac{\pi}{2}, \pi$.	**4**	**CO2**
Q 5	Determine whether the system of functions $\{1, x^2, x^3\}$ satisfies the Haar condition on the interval $[-1,1]$.	**4**	**CO4**

SECTION B

Answer all questions. There is an internal choice in Q9.

Q 6	Let X be a normed linear space and $Y \subset X$ a subspace of it and $x \in X$. Prove that there always exist $y^* \in Y$ such that $\|x - y^*\| = \min_{y \in Y} \|x - y\|$.	**10**	**CO1**

Q 7	Show that $B_n f \longrightarrow f$ for $f(x) = x^3$ where $B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) {}^{n}C_k \, x^k (1-x)^{n-k}$	**10**	**CO3**
Q8	Find the least positive integer n such that $\left\|\|B_n f - f\|\right\| < 10^{-3}$ for $f(x) = x^2$, where $B_n f$ is the Bernstein polynomial for $f \in \mathcal{C}[0,1]$.	**10**	**CO3**
Q 9	Find a polygonal function $p(x) \in S_2(x)$ where $S_2(x)$ is the space of polygonal functions defined for $x_0 \leq x_1 \leq x_2$ that approximate the function $f(x) = x^2$ on $[x_0, x_2]$ where $x_0 = 0,\ x_2 = 1$ and x_1 divides the interval in the ratio $1:3$. **OR** Let $f \in \mathcal{C}[a,b], 0 < a < b < 1$. Prove that there exists a sequence of polynomials with integer coefficients converging uniformly to f.	**10**	**CO2**
SECTION-C **Answer all questions. There is an internal choice in Q9.**			
Q10	Find the least n such that $\sup_{x \in [0,2]} \left\|\|f(x) - p(x)\|\right\| < 10^{-2}$ where $f(x) = x^2$ and $p(x)$ is the function defined as $$p(x) = \begin{cases} ax + b, & \text{if } 0 \leq x \leq \frac{1}{n} \\ cx + d, & \text{if } \frac{1}{n} \leq x \leq 2 \end{cases}$$	**20**	**CO2**
Q11	Find the projection of $f(x) = e^x$ on to the subspace $\mathcal{W}$ of $\mathcal{C}[0,1]$ that is spanned by $\{1, x\}$. Determine the sum squared error when approximating by this projection. **OR** a) Find the second order Fourier approximation of $E(t) = t + 1$. Provide a general formula for n^{th} order Fourier approximation of $E(t)$. b) Determine the first order Maclaurin's polynomial of $f(x) = e^x$ and obtain the sum squared error when approximating by this polynomial.	**20**	**CO4**